EXERCISE – III

HINTS & SOLUTIONS

Sol.1 Directrix $2x + y - 1 = 0$ Let p (h, k)

$S(1, 1)$ $e = \sqrt{3}$

$SP = ePM$

$SP^2 = e^2 PM^2$

$$(h-1)^2 + (k-1)^2 = 3\left|\frac{2h+k-1}{\sqrt{5}}\right|^2$$

$\Rightarrow 7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$

L.R. = 2e × (Distance between focus and directrix)

$$= 2\sqrt{3} \times \left|\frac{2+1-1}{\sqrt{5}}\right| = \frac{\sqrt{48}}{\sqrt{5}} = \frac{\sqrt{48}}{\sqrt{5}}$$

Sol.2 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ...(1)

$7x + 13y - 87 = 0$

$5x - 8y + 7 = 0$

POI (5, 4)

Equation (1) pass through (5, 4)

$\dfrac{25}{a^2} - \dfrac{16}{b^2} = 1$ ...(2)

and $\dfrac{2b^2}{a} = \dfrac{32\sqrt{3}}{5}$ ...(3)

From (2) & (3)

$a^2 = \dfrac{25}{2}$, $b^2 = 16$

Sol.3 $\dfrac{x^2}{100} - \dfrac{y^2}{25} = 1$

(i) $e^2 = 1 + \dfrac{b^2}{a^2} = 1 + \dfrac{25}{100} = \dfrac{125}{100}$

$e = \sqrt{\dfrac{125}{100}} = \dfrac{\sqrt{5}}{2}$

(ii) SA. S′A = (ae – a) . (ae + a)

$= a^2 (e^2 - 1) = 100\left(\dfrac{5}{4} - 1\right) = 25$

Sol.4 $16x^2 + 32x - 9y^2 + 36y - 164 = 0$

$16(x^2 + 2x + 1) - 16 - 9(y^2 - 4y + 4) + 36 - 164 = 0$

$16(x-1)^2 - 9(y^2 - 2) = 144$

$\dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{16} = 1$

Centre (–1, 2) ; $e^2 = 1 + \dfrac{16}{9} \Rightarrow e = \dfrac{5}{3}$

Foc ; $x + 1 = \pm 5$ and $y - 2 = 0$

foci (4, 2) & (–6, 2)

Directrix $x + 1 = \pm \dfrac{a}{e}$

$x + 1 = \pm \dfrac{9}{5}$

$\Rightarrow 5x - 4 = 0$ and $5x + 14 = 0$

L.R. $= \dfrac{2b^2}{a} = \dfrac{32}{3}$

$\ell(TA) = 2a = 6$

$\ell(CA) = 2b = 8$

Axis $x + 1 = 0$ and $y - 2 = 0$

Sol.5 $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$

TAngent : $y = mx \pm \sqrt{a^2m^2 - b^2}$

$m = -1$

$a^2 = 36$, $b^2 = 9$

$y = -x \pm \sqrt{27}$

$y = -x \pm 3\sqrt{3}$

Sol.6 $\dfrac{3x^2}{25} - \dfrac{2y^2}{25} = 1$

$y = mx \pm \sqrt{a^2m^2 - b^2}$

(0, 5/2)

$\dfrac{5}{2} = \pm\sqrt{a^2m^2 - b^2}$

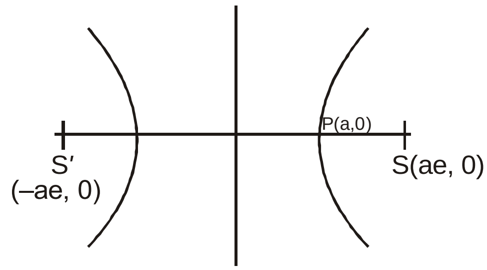
$\dfrac{25}{4} = a^2m^2 - b^2$

$$\frac{25}{4} = \frac{25}{3}m^2 - \frac{25}{2} \Rightarrow m = \pm\frac{3}{2}$$

Equation of tangents

$3x + 2y - 5 = 0$ and $3x - 2y + 5 = 0$

Sol.7 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



Let P(a, 0)

SP. S′P = (ae – a) (ae + a)

$= a^2(e^2 - 1) = b^2$

$CP^2 - a^2 + b^2 = a^2 - a^2 + b^2 = b^2$

L.H.S. = R.H.S.

Sol.8 Equation of chord connecting the points $(a\sec\theta_1, b\tan\theta_1)$ and $(a\sec\theta_2, b\tan\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$= \cos\left(\frac{\theta_1+\theta_2}{2}\right) \qquad ...(1)$$

If it passes thorugh (ae, 0)

We have $e\,.\cos\left(\dfrac{\theta_1-\theta_2}{2}\right) = \cos\left(\dfrac{\theta_1+\theta_2}{2}\right)$

$$\Rightarrow e = \frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} = \frac{1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}{1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}$$

$$\Rightarrow \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1-e}{1+e}$$

Sol.9 $3x^2 - 2y^2 = 6$

$$\frac{x^2}{2} - \frac{y^2}{3} = 1$$

$$y = mx \pm \sqrt{2m^2-3}$$

(α, β)

$(\beta - m\alpha)^2 = 2m^2 - 3$

$\beta^2 + m^2\alpha^2 - 2\alpha\beta m - 2m^2 + 3 \;\; 0$

$$m^2(\alpha^2 - 2) - 2\alpha\beta m + 3 + \beta^2 = 0 \quad \begin{matrix} \nearrow m_1 \\ \searrow m_2 \end{matrix}$$

$$m_1 \,.\, m_2 = \frac{3+\beta^2}{\alpha^2-2}$$

$$2 = \frac{3+\beta^2}{\alpha^2-2}$$

$2\alpha^2 - 4 = 3 + \beta^2$

$\beta^2 = 2\alpha^2 - 7$

Sol.10 Let CP = r_1 and angle to x-axis will be θ

P = $(r_1\cos\theta, r_1\sin\theta)$ and P lies on that hyperbola

$$r_1^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1$$

Replace θ by $90° + \theta$

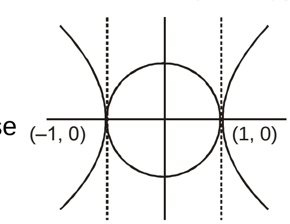
$$r_2^2\left(\frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}\right) = 1$$

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = \cos^2\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \sin^2\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Sol.11 $e = \dfrac{1}{2}$; Focus S $\left(\dfrac{1}{2}, 1\right)$

common tangent of circle and rectangular hyp. is x = 1 or x – 1 = 0



Let P(h, k)

m : x – 1 = 0

By Basic definition of ellipse

PS = ePM

$PS^2 = e^2\, PM^2$

$$(h - 1/2)^2 + (k-1)^2 = \frac{1}{4}\left|\frac{h-1}{1}\right|^2$$

after solving $\dfrac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}} + \dfrac{(y-1)^2}{\frac{1}{12}} = 1$

Sol.12 Let a point P(a secθ, b tanθ)

tangent : $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

$$A\left(\theta, \frac{-b}{\tan\theta}\right)$$

Normal : $ax\cos\theta + by\cot\theta = a^2e^2$

$$B\left(0, \frac{a^2e^2}{b\cot\theta}\right)$$

Equation of circle AB as diameter

$$x^2 + \left(y + \frac{b}{\tan\theta}\right)\left(y - \frac{a^2e^2}{b\cot\theta}\right) = 0$$

It is passes through (ae, 0) which is focus of hyperbola.

Sol.13 Any normal at the point P(θ) is

$ax\cos\theta + by\cot\theta = a^2 + b^2$ ...(1)

Any line through centre (0, 0) and perpendicular to Equation (1) is

$bx\cot\theta - ay\cos\theta = 0$

$\therefore \sin\theta = \frac{bx}{ay}$

$\therefore \cos\theta = \sqrt{\frac{a^2y^2 - b^2x^2}{ay}}$

$\cot\theta = \frac{\sqrt{a^2y^2 - b^2x^2}}{bx}$

Substituting the values of cosθ and cotθ is ...(4) we get

$$\sqrt{a^2y^2 - b^2x^2}\left[ax.\frac{1}{ay} + by\frac{1}{bx}\right] = a^2 + b^2$$

$(a^2y^2 - b^2x^2)(x^2 + y^2)^2 = (a^2 + b^2)x^2y^2$

Sol.14 Let the point P (a secθ, b tanθ)

Normal at point P

$ax\cos\theta + by\cot\theta = a^2e^2$

at x-axis y = 0

$G\left(\frac{ae^2}{\cos\theta}, 0\right)$ or G ($ae^2\sec\theta$, 0)

We have to prove that SG = eSP

$SG = ae - ae^2\sec\theta$

We also know that

SP = ePM m : directrix

$= e|a\sec\theta - a/e|$ $x = \frac{a}{e}$

$eSP = e^2|a\sec\theta - a/e|$

$= |ae - ae^2\sec\theta|$ so SG = eSP

Sol.15 If (h, k) be the mid-point of the chord of the Hyperbola $x^2 - y^2 = a^2$ than its equation by $T = S_1$ is

$hx - ky = h^2 - k^2$ ...(1)

But Equation (1) is Normal to the Hyp. its equation is

$x\cos\theta + y\cot\theta = 2a$ ...(2)

(1) and (2) are same

$$\frac{h}{\cos\theta} = \frac{-k}{\cot\theta} = \frac{h^2 - k^2}{2a}$$

$\therefore \sec\theta = \frac{h^2 - k^2}{2ah}$ and $\tan\theta = \frac{h^2 - k^2}{-2ak}$

$\sec^2\theta - \tan^2\theta = 1$

$$\therefore \frac{(h^2 - k^2)^2}{4a^2} . \left(\frac{1}{h^2} - \frac{1}{k^2}\right) = 1$$

Locus

$(y^2 - x^2)^3 = 4a^2x^2y^2$

Sol.16 Chord joining two points P(θ) and Q(ϕ)

$$x\cos\left(\frac{\theta - \phi}{2}\right) - y\sin\left(\frac{\theta + \phi}{2}\right) = a\cos\left(\frac{\theta + \phi}{2}\right)$$

$$\text{Slope} = \frac{\cos\left(\frac{\theta - \phi}{2}\right)}{\sin\left(\frac{\theta + \phi}{2}\right)}$$

Normal at P(θ)

So slope of normal = $-\sin\theta$

$$-\sin\theta = \frac{\cos\left(\frac{\theta - \phi}{2}\right)}{\sin\left(\frac{\theta + \phi}{2}\right)}$$

After solving

$\tan\phi = \tan\theta(4\sec^2\theta - 1)$

Sol.17 Focii S(ae, 0) S′(–ae, 0)
circle with diameter SS′
$(x – ae)(x + ae) + y^2 = 0$
$x^2 + y^2 = a^2e^2$ ...(1)
If (h, k) be the pole of the chord which touches Equation (1) then its equation is the polar of (h,k) w.r.t. Hyp.

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1 \qquad ...(2)$$

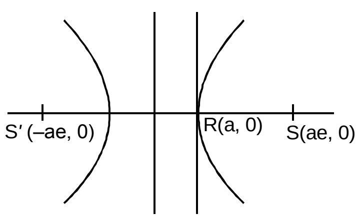
Line (2) Touches the circle (1)

$$\frac{1}{\sqrt{\frac{h^2}{a^4}+\frac{k^2}{b^4}}} = ae$$

$$\Rightarrow \frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{1}{a^2e^2} = \frac{1}{a^2+b^2}$$

Locus $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2+b^2}$

Sol.18 $(RS + RS')^2 = (ae – a + a + ae)^2$
$= 4a^2e^2$



RHS $= 4a^2\left(1+\frac{b^2}{p^2}\right)$

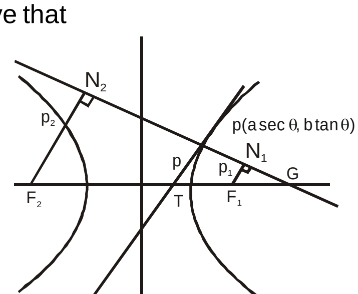
$= 4a^2\left(1+\frac{b^2}{a^2}\right)$

$= 4(a^2 + b^2) = 4a^2e^2$
LHS. = RHS.

Sol.19 We wish to prove that

$$p = \frac{2p_1p_2}{p_1+p_2}$$

or $\frac{2}{p} = \frac{1}{p_1} + \frac{1}{p_2}$



Tangent :

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

Normal

$$\frac{a^2x}{a\sec\theta} + \frac{b^2y}{b\tan\theta} = a^2e^2$$

T(a cosθ, 0) F_1 (ae, 0)
G $(ae^2 \sec\theta, 0)$
$\Delta TPG \sim \Delta F_1N_1G$

$$\frac{p}{p_1} = \frac{TG}{F_1G} = \frac{ae^2\sec\theta - a\cos\theta}{ae^2\sec\theta - ae}$$

$$\frac{p}{p_1} = \frac{e^2\sec\theta - \cos\theta}{e(e\sec\theta - 1)}$$

$$\frac{p}{p_1} = \frac{e+\cos\theta}{e}$$

Similarly

$$\frac{p}{p_2} = \frac{TG}{F_2G} = \frac{e-\cos\theta}{e}$$

$$\frac{p}{p_1} + \frac{p}{p_2} = 2$$

$$\frac{2}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

Sol.20

Given $2ae = 2\sqrt{13}$

$\Rightarrow a_1e_1 = \sqrt{13}$ & $a_2e_2 = \sqrt{13}$

$a_2 – a_1 = 4$ $\qquad a_1 \Rightarrow$ Hyper.

& $\frac{e_2}{e_1} = \frac{3}{7}$ $\qquad a_2 \Rightarrow$ ellipse

$$\frac{a_1}{a_2}\cdot\frac{e_1}{e_2} = 1 \Rightarrow \frac{a_1}{a_2} = \frac{3}{7}$$

$a_1 = 3$ & $a_2 = 7$

$e_1 = \frac{\sqrt{13}}{3}$ & $e_2 = \frac{\sqrt{13}}{7}$

$b_1^2 = 4$ & $b_2^2 = 36$

ellipse : $\frac{x^2}{49} + \frac{y^2}{36} = 1$

Hyper : $\frac{x^2}{9} - \frac{y^2}{4} = 1$